Vertical Specialization and the Interdependence of Nations*

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Abstract

This paper develops an elementary theory of trade with sequential production. In spite of its extreme simplicity, our theory is consistent with a number of stylized facts and able to deliver a rich set of predictions regarding how vertical specialization shapes the interdependence of nations. Among other things, we show that TFP growth in any country participating in a global supply chain leads all its trading partners to "move up" the chain, and in turn, always increases inequality between countries at the bottom of the chain. Our results point towards the importance of modelling the sequential nature of production for understanding the consequences of technological changes in developing and developed countries on their trading partners worldwide.

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"One man draws out the wire, another straightens it, a third cuts it, a fourth points it, a fifth grinds it at the top for receiving the head; to make the head requires two or three distinct operations; to put it on is a peculiar business; to whiten the pins is another; it is even a trade by itself to put them into the paper; and the important business of making a pin is, in this manner, divided into about 18 distinct operations, which, in some manufactories, are all performed by distinct hands, though in some others the same man will sometimes perform two or three of them," Adam Smith (1776).

1 Introduction

Most production processes consist of a large number of sequential stages. In this regard the production of pins in late eighteenth century England is no different from today's production of tee-shirts, cars, computers, or semi-conductors. Today, however, production processes increasingly involve vertical supply chains spanning multiple countries, with each country specializing in particular stages of a good's production sequence, a phenomenon which Hummels, Ishii, and Yi (2001) refer to as "vertical specialization."

This global phenomenon has attracted a lot of attention among policy makers, business leaders, and trade economists alike. On the academic side of this debate, a large literature has emerged to investigate how the possibility to fragment production processes across borders may affect the volume, pattern, and consequences of international trade; see e.g. Feenstra and Hanson (1996), Yi (2003), and Grossman and Rossi-Hansberg (2008). In this paper, we propose to take a first look at a distinct, but equally important question: Conditional on production processes being fragmented across borders, how do structural changes in one country affect other countries located at different stages of the same supply chain? In other words, how does vertical specialization shape the "interdependence of nations?"

Theoretically, deriving sharp and intuitive comparative static predictions in general equilibrium models with an arbitrary number of goods and countries is notoriously difficult.[1] In order to make progress on this question, we therefore start by developing an elementary theory of trade with sequential production. We consider a world economy with multiple countries, one factor of production (labor), and one final good. Production is sequential and subject to mistakes, as in Sobel (1992) and Kremer (1993). Production of the final good requires a continuum of intermediate stages. At each of these stages, production of one unit

[1] Ethier (1984) offers a review of theoretical results in high-dimensional trade models.

of an intermediate good requires one unit of labor and one unit of the intermediate good produced in the previous stage. Mistakes occur along the supply chain at a constant Poisson rate, which is an exogenous technological characteristic of a country. When a mistake occurs at some stage, the intermediate good is entirely lost. By these stark assumptions, we aim to capture the more general idea that because of less skilled workers, worse infrastructure, or inferior contractual enforcement, both costly defects and delays in production are more likely in some countries than others.

In spite of its extreme simplicity, our theory of trade with sequential production is consistent with a number of stylized facts and able to deliver a rich set of predictions regarding how vertical specialization shapes the interdependence of nations. Among other things, we show that TFP growth in any country participating in a global supply chain leads all its trading partners to "move up" the chain, and in turn, always increases inequality between countries at the bottom of the chain. Our results point towards the importance of modelling the sequential nature of production for understanding the consequences of technological changes in developing and developed countries on their trading partners worldwide.

Section 3 describes the properties of the free trade equilibrium in our basic environment. Although our model allows for any finite number of countries, the unique free trade equilibrium is fully characterized by a simple system of first-order non-linear difference equations. This system can be solved recursively by first determining the assignment of countries to different stages of production and then computing the wages and export prices sustaining that allocation as an equilibrium outcome. In our model, the free trade equilibrium always exhibits vertical specialization: countries with higher TFP, which have a lower probability of making mistakes, specialize in later stages of production, where mistakes are more costly. Compared to standard Ricardian models, absolute rather than relative productivity differences are a source of comparative advantage among nations.

As mentioned before, our theory of trade with sequential production is consistent with a number of stylized facts. First, since less productive countries produce and export at earlier stages of production, our model predicts that poor countries have higher shares of primary production in value added.[2] Second, since rich countries tend to specialize in later stages of production while poor countries tend to specialize in earlier stages, our model implies that rich countries tend to trade relatively more with other rich countries (from whom they import their intermediates and to whom they export their output) while poor countries tend to trade relatively more with other poor countries, as documented by Hallak (2010). Third,

[2] This observation was originally made by Kremer (1993) in the context of his closed-economy model.

since intermediate goods produced in later stages have higher prices and countries producing in these stages have higher wages, our model implies that rich countries both tend to import goods with higher unit values, as documented by Hallak (2006), and to export goods with higher unit values, as documented by Schott (2004), Hummels and Klenow (2005), and Hallak and Schott (2010).[3]

Section 4 presents two comparative static exercises: population growth and TFP growth. While interesting in its own right, our first comparative static exercise will also help us shed light on how the sequential nature of production affects the impact of technological changes around the world. In a standard Ricardian model without sequential production, population and TFP growth in one country should have the exact same effects on its trading partners. In our model, they will not. In fact, at the bottom of the chain, population and TFP growth in one country will have the exact opposite effects on its trading partners.

After population growth in one country, market clearing requires this country to perform more stages and all other countries to perform fewer stages. Hence, countries at the bottom of the chain move further down, whereas countries at the top move further up in response to foreign population growth. At the bottom of the chain, these changes in the pattern of vertical specialization are accompanied by declining inequality between nations. By moving down and performing fewer stages, countries import and export intermediate goods with higher labor intensities, i.e., goods with higher labor cost shares. Since poor countries are those with a comparative advantage in these goods, their wages increase relative to the wages of richer countries. By contrast, population growth has a non-monotonic effect on inequality between nations at the top of the chain, reflecting two conflicting forces. On the one hand, countries are moving up, which tends to lower the labor cost shares of the intermediate goods that they trade. On the other hand, countries are performing fewer stages, which tends to raise labor cost shares. Since poorer countries have a comparative advantage in stages with higher labor intensities, the first force increases inequality between nations, whereas the second force, like at the bottom of the chain, decreases it.

The consequences of TFP growth are quite different. In this case, market clearing requires all countries to perform more stages at the bottom of the chain, but fewer stages at the top. Hence, all countries move up the supply chain in response to foreign productivity growth. Changes in the pattern of vertical specialization at the bottom of the chain are, therefore,

[3]Following Linder (1961), a large theoretical literature has offered demand-based explanations of such patterns emphasizing the role of non-homothetic preferences; see e.g. Markusen (1986), Flam and Helpman (1987), Bergstrand (1990), Stokey (1991), Murphy and Shleifer (1997) Matsuyama (2000), Fieler (2008), and more recently, Fajgelbaum, Grossman, and Helpman (2009). We come back to this literature in Section 3.3.

the exact opposite of what they were after population growth. Using the same logic as in our first comparative static exercise, we can then show that changes in inequality between countries at the bottom of the chain are the exact opposite as well. By moving up and performing more stages, countries import and export intermediate goods with lower labor intensities. Since these are the goods in which richer countries have a comparative advantage, inequality between countries increases at the bottom of the chain. Finally, like in the case of population growth, TFP growth has a non-monotonic effect on inequality at the top of the supply chain, reflecting the fact that these countries are both moving up and performing fewer stages.

Section 5 discusses how more realistic features of global supply chains may be incorporated within our simple theoretical framework. Our first extension allows for the coexistence of multiple supply chains. This generalization of our model nests the standard Ricardian model, when each chain is located in a single country. Our second extension allows for heterogeneity in failure rates across different stages, thereby recognizing that intermediate goods may differ by more than the order in which they are performed. Put together, our first and second extensions give us a simple way to introduce final assembly into our model and to explain why it tends to occur in countries with low TFPs in practice. Our final extension introduces iceberg trade costs, which naturally rationalizes why vertical specialization tends to be a regional phenomenon in practice (e.g. within NAFTA, the EU, or East Asia).

Our paper is related to several strands of the literature. First, we draw some ideas from the literature on hierarchies in closed-economy (and mostly partial-equilibrium) models. Important early contributions include Lucas (1978), Rosen (1982), Sobel (1992), Kremer (1993), Garicano (2000) and Garicano and Rossi-Hansberg (2006). As in Sobel (1992) and Kremer (1993), we focus on an environment in which production is sequential and subject to mistakes, though we do so in a general equilibrium, open-economy setup. Models of hierarchies have been applied to the study of international trade issues before, but with very different goals in mind. For instance, Antràs, Garicano, and Rossi-Hansberg (2006) use the knowledge economy model developed by Garicano (2000) to study the matching of agents with heterogeneous abilities across borders and its consequences for within-country inequality. Instead, countries are populated by homogeneous workers in our model.[4]

In terms of techniques, our paper is also related to a growing literature using assignment or matching models in an international context; see, for example, Grossman and Maggi (2000),

[4] Other examples of trade papers using hierearchy models to study within-country inequality include Kremer and Maskin (2006), Sly (2010), Monte (2010), and Sampson (2010).

Grossman (2004), Yeaple (2005), Ohnsorge and Trefler (2007), Blanchard and Willmann (2008), Nocke and Yeaple (2008), Costinot (2009), and Costinot and Vogel (2010). Here, like in some of our earlier work, we exploit the fact that the assignment of countries to stages of production exhibits positive assortative matching—i.e., more productive countries are assigned to later stages of production—in order to generate strong and intuitive comparative static predictions in an environment with a large number of goods and countries.

In terms of focus, our paper is motivated by the recent literature documenting the importance of vertical specialization in world trade. On the empirical side, this literature builds on the influential work of Hummels, Rappoport, and Yi (1998), Hummels, Ishii, and Yi (2001), and Hanson, Mataloni, and Slaughter (2005).[5] Our focus on how vertical specialization shapes the interdependence of nations is also related to the work of Kose and Yi (2001, 2006), Burstein, Kurz, and Tesar (2008), and Bergin, Feenstra, and Hanson (2009) who study how production sharing affects the transmission of shocks at business cycle frequency.

On the theoretical side, the literature on fragmentation is large and diverse. A non-exhaustive list of papers on this topic include Dixit and Grossman (1982), Sanyal (1983), Jones and Kierzkowski (1990, 2001), Feenstra and Hanson (1996), Arndt (1997), Deardorff (2001a,b), Egger and Falkinger (2003), Yi (2003, 2010), Kohler (2004), Baldwin and Robert-Nicoud (2007), Rodríguez-Clare (2010), and Grossman and Rossi-Hansberg (2008, 2010); see Antràs and Rossi-Hansberg (2009) for a recent overview. Among the previous papers, our theoretical framework is most closely related to Dixit and Grossman (1982), Sanyal (1983), and Yi (2003, 2010) who also consider environments with sequential production. None of these papers, however, investigate how structural changes in one country may differentially impact other countries located at different stages of the same supply chain. This is the main focus of our analysis.

2 A Simple Trade Model with Sequential Production

We consider a world economy with multiple countries, indexed by $c \in \mathcal{C} \equiv \{1, ..., C\}$, one factor of production, labor, and one final good. Labor is inelastically supplied and immobile across countries. L_c and w_c denote the endowment of labor and wage in country c, respectively. Production of the final good is sequential and subject to mistakes. To produce the final good, a continuum of stages $s \in \mathcal{S} \equiv (0, 1]$ must be performed. At each stage, producing one unit of intermediate good requires one unit of the intermediate good produced in the

[5] See also Johnson and Noregua (2010) for a generalization of Hummels, Ishii, and Yi's (2001) methodology.

previous stage and one unit of labor. For expositional purposes, we assume that "intermediate good 0" is in infinite supply and has zero price.[6] "Intermediate good 1" corresponds to the unique final good mentioned before. Mistakes occur along the supply chain at a constant Poisson rate, $\lambda_c > 0$, which is an exogenous technological characteristic of a country. It measures total factor productivity (TFP) at each stage of the production process. When a mistake occurs on a unit of intermediate good at some stage, that intermediate good is entirely lost. Formally, if a firm from country c combines $q(s)$ units of intermediate good s with $q(s)ds$ units of labor, its output of intermediate good $s + ds$ is given by

$$q(s + ds) = (1 - \lambda_c ds)\, q(s)\,. \tag{1}$$

For technical reasons, we further assume that if a firm produces intermediate good $s+ds$, then it necessarily produces a positive measure of intermediate goods around that stage.[7] This implies that each unit of the final good is produced by a finite, though possibly arbitrarily large number of firms. Countries are ordered such that λ_c is strictly decreasing in c. Thus countries with a higher index c have higher total factor productivity. All markets are perfectly competitive and all goods are freely traded. $p(s)$ denotes the world price of intermediate good s. We use the final good as our numeraire, $p(1) = 1$.

3 Free Trade Equilibrium

3.1 Definition

In a free trade equilibrium, all firms maximize their profits taking world prices as given and all markets clear. Profit maximization requires that for all $c \in \mathcal{C}$,

$$\begin{aligned} &p(s + ds) \leq (1 + \lambda_c ds)\, p(s) + w_c ds, \\ &p(s + ds) = (1 + \lambda_c ds)\, p(s) + w_c ds, \text{ if } Q_c(s') > 0 \text{ for all } s' \in (s, s + ds], \end{aligned} \tag{2}$$

[6] Alternatively, one could assume that "intermediate good 0" can be produced using labor only. In this situation, the price of "intermediate good 0" would also be zero since only a measure zero of workers would be required to perform this measure-zero set of stages. Assuming that "intermediate good 0" is in infinite supply allows us to avoid discussions of which country should produce this good. Such considerations are, of course, irrelevant for any of our results.

[7] Formally, for any intermediate good $s + ds$, we assume the existence of $s_2 \geq s + ds > s_1$ such that if $q(s + ds) > 0$, then $q(s') > 0$ for all $s' \in (s_1, s_2]$.

where $Q_c(s')$ denotes total output at stage s' in country c. Condition (2) states that the price of intermediate good $s+ds$ must be weakly less than its unit cost of production, with equality if intermediate good $s+ds$ is actually produced by a firm from country c. To see this, note that the production of one unit of intermediate good $s+ds$ requires $1/(1-\lambda_c ds)$ units of intermediate good s as well as labor for all intermediate stages in $(s, s+ds]$. Thus the unit cost of production of intermediate good $s+ds$ is given by $[p(s)+w_c ds]/(1-\lambda_c ds)$ which is equal to $(1+\lambda_c ds)p(s)+w_c ds$ since ds is infinitesimal. Good and labor market clearing further require that

$$\sum_{c=1}^{C} Q_c(s_2) - \sum_{c=1}^{C} Q_c(s_1) = -\int_{s_1}^{s_2} \sum_{c=1}^{C} \lambda_c Q_c(s)\, ds, \text{ for all } s_1 \leq s_2, \quad (3)$$

$$\int_0^S Q_c(s)\, ds = L_c, \text{ for all } c \in \mathcal{C}, \quad (4)$$

Equation (3) states that the change in the world supply of intermediate goods between stages s_1 and s_2 must be equal to the amount of intermediate goods lost due to mistakes in all countries between these two stages. Equation (4) states that the total amount of labor used across all stages must be equal to the total supply of labor in country c. In the rest of this paper, we formally define a free trade equilibrium as follows.

Definition 1 *A free trade equilibrium corresponds to output levels $Q_c(\cdot): \mathcal{S} \longrightarrow \mathbb{R}^+$ for all $c \in \mathcal{C}$, wages $w_c \in \mathbb{R}^+$ for all $c \in \mathcal{C}$, and intermediate good prices $p(\cdot): \mathcal{S} \longrightarrow \mathbb{R}^+$ such that conditions* (2)-(4) *hold.*

3.2 Existence and Uniqueness

We first characterize the pattern of international specialization in any free trade equilibrium.

Lemma 1 *In any free trade equilibrium, there exists a sequence of stages $S_0 \equiv 0 < S_1 < ... < S_C = 1$ such that for all $s \in \mathcal{S}$ and $c \in \mathcal{C}$, $Q_c(s) > 0$ if and only if $s \in (S_{c-1}, S_c]$.*

According to Lemma 1, there is vertical specialization in any free trade equilibrium with more productive countries producing and exporting at later stages of production. The intuition behind Lemma 1 can be understood in two ways. One possibility is to note that since new intermediate goods require both intermediate goods produced in previous stages and labor, prices must be increasing along the supply chain. Thus intermediate goods produced at later stages are less labor intensive, which makes them *relatively* cheaper to produce in countries with higher wages. In our model these are the countries with an *absolute* advantage in all goods. Alternatively, one can look at Lemma 1 through the lens of the hierarchy literature; see e.g. Lucas (1978), Rosen (1982), and Garicano (2000). Since countries that are producing at later stages can leverage their productivity on larger amounts of inputs, efficiency requires countries to be more productive at the top. A result similar to Lemma 1 in an environment with a discrete number of stages can also be found in Sobel (1992) and Kremer (1993).

We refer to the vector $(S_1, ..., S_C)$ as the "pattern of vertical specialization" and denote by $Q_c \equiv Q_c(S_c)$ the total amount of intermediate good S_c produced and exported by country c. Using the previous notation, the pattern of vertical specialization and export levels can be jointly characterized as follows.

Lemma 2 *In any free trade equilibrium, the pattern of vertical specialization and export levels satisfy the following system of first-order non-linear difference equations:*

$$\begin{aligned} S_c &= S_{c-1} - \left(\frac{1}{\lambda_c}\right) \ln\left(1 - \frac{\lambda_c L_c}{Q_{c-1}}\right), \textit{ for all } c \in \mathcal{C}, & (5) \\ Q_c &= e^{-\lambda_c(S_c - S_{c-1})} Q_{c-1}, \textit{ for all } c \in \mathcal{C}, & (6) \end{aligned}$$

with boundary conditions $S_0 = 0$ and $S_C = 1$.

Lemma 2 derives from the goods and labor market clearing conditions (3) and (4). Equation (5) reflects the fact that the exogenous supply of labor in country c must be equal to the amount of labor demanded to perform all stages from S_{c-1} to S_c. This amount of labor depends both on the rate of mistakes, λ_c, as well as the total amount Q_{c-1} of intermediate good S_{c-1} imported from country $c-1$. Equation (6) reflects the fact that intermediate goods get lost at a constant rate at each stage when produced in country c.

In the rest of this paper, we refer to the vector of wages $(w_1, ..., w_C)$ as the "world income distribution" and to $p_c \equiv p(S_c)$ as the price of country c's exports (which is also the price of country $c+1$'s imports under free trade). Let $N_c \equiv S_c - S_{c-1}$ denote the measure of stages performed by country c within the supply chain. In the next lemma, we show that the measures of stages being performed in all countries $(N_1, ..., N_C)$ entirely summarize how changes in the pattern of vertical specialization affect the world income distribution.

Lemma 3 *In any free trade equilibrium, the world income distribution and export prices* *satisfy the following system of first-order linear difference equations:*

$$w_{c+1} = w_c + (\lambda_c - \lambda_{c+1})\, p_c, \text{ for all } c < C, \tag{7}$$

$$p_c = e^{\lambda_c N_c} p_{c-1} + \left(e^{\lambda_c N_c} - 1\right)(w_c/\lambda_c)\,, \text{ for all } c \in \mathcal{C}, \tag{8}$$

with boundary conditions $p_0 = 0$ *and* $p_C = 1$*.*

Lemma 3 derives from the zero-profit condition (2). Equation (7) reflects the fact that for the "cut-off" good, S_c, the unit cost of production in country c, $(1 + \lambda_c ds)\, p_c + w_c ds$, must be equal to the unit cost of production in country $c+1$, $(1 + \lambda_{c+1} ds)\, p_c + w_{c+1} ds$. Equation (8) directly derives from the zero-profit condition (2) and the definition of N_c and p_c. It illustrates the fact that the price of the last intermediate good produced by country c depends on the price of the intermediate good imported from country $c-1$ as well as the total labor cost in country c.

Combining Lemmas 1-3, we can establish the existence of a unique free trade equilibrium and characterize its main properties.

Proposition 1 *There exists a unique free trade equilibrium. In this equilibrium, the pattern of vertical specialization and export levels are given by equations* (5) *and* (6)*, and the world income distribution and export prices are given by equations* (7) *and* (8)*.*

The proof of Proposition 1 formally proceeds in two steps. First, we use Lemma 2 to construct the unique pattern of vertical specialization and vector of export levels. In equations (5) and (6), we have one degree of freedom, Q_0, which corresponds to total input used at the initial stage of production and can be set to satisfy the final boundary condition $S_C = 1$. Once $(S_1, ..., S_C)$ and $(Q_0, ..., Q_{C-1})$ have been determined, all other output levels can be computed using equation (1) and Lemma 1. Second, we use Lemma 3 together with the equilibrium measure of stages computed before, $(N_1, ..., N_C)$, to characterize the unique world income distribution and vector of export prices. In equations (7) and (8), we still have one degree of freedom, w_1, which can be used to satisfy the other final boundary condition, $p_C = 1$. Finally, once $(w_1, ..., w_C)$ and $(p_1, ..., p_C)$ have been determined, all other prices can be computed using the zero-profit condition (2) and Lemma 1.

3.3 Discussion

In spite of its extreme simplicity, our theory of trade with sequential production delivers a rich set of predictions. First, since less productive countries produce and export at earlier stages of production, our model predicts that poor countries have higher shares of primary production in value added. Second, since rich countries tend to specialize in later stages of production while poor countries tend to specialize in earlier stages, our model implies that rich countries tend to trade relatively more with other rich countries (from whom they import their intermediates and to whom they export their output) while poor countries tend to trade relatively more with other poor countries, as documented by Hallak (2010). Third, since intermediate goods produced in later stages have higher prices and countries producing in these stages have higher wages, our model implies that rich countries both tend to import goods with higher unit values, as documented by Hallak (2006), and to export goods with higher unit values, as documented by Schott (2004), Hummels and Klenow (2005), and Hallak and Schott (2010).

Following Linder (1961), the last two previous stylized facts have traditionally been rationalized using non-homothetic preferences; see e.g. Markusen (1986), Flam and Helpman (1987), Bergstrand (1990), Stokey (1991), Murphy and Shleifer (1997) Matsuyama (2000), Fieler (2008), and Fajgelbaum, Grossman, and Helpman (2009). The common starting point of the previous papers is that rich countries' preferences are skewed towards high quality goods, so they tend to import goods with higher unit values. Under the assumption that rich countries are also relatively better at producing high quality goods, these models can further explain why rich countries tend to export goods with higher unit values and why countries with similar levels of GDP per capita tend to trade more with each other.[8]

The complementary explanation offered by our elementary theory of trade with sequential production is very different. According to our model, countries with similar per-capita incomes are more likely to trade with one another because they specialize in nearby regions of the same supply chain. Similarly, countries with higher levels of GDP per capita tend to have higher unit values of imports and exports because they specialize in higher stages in the supply chain, for which inputs and outputs are more costly.

[8] In Fajgelbaum, Grossman, and Helpman (2009), such predictions are obtained in the absence of any exogenous relative productivity differences. In their model, a higher relative demand for high-quality goods translates into a higher relative supply of these goods through a "home-market" effect.

4 Comparative Statics

4.1 Definitions

Before turning to our two main comparative static exercises, it is useful to introduce some formal definitions describing the changes in the pattern of vertical specialization and the world income distribution in which we will be interested.

Definition 2 *Let* $(S'_1, ..., S'_C)$ *denote the pattern of vertical specialization in a counterfactual free trade equilibrium. A country* $c \in \mathcal{C}$ *is moving up (resp. down) the supply chain relative to the initial free trade equilibrium if* $S'_c \geq S_c$ *and* $S'_{c-1} \geq S_{c-1}$ *(resp.* $S'_c \leq S_c$ *and* $S'_{c-1} \leq S_{c-1}$*).*

Formally, a country is moving up or down the supply chain if we can rank the set of stages that it performs in the initial and counterfactual free trade equilibria in terms of the strong set order.[9] This simple mathematical notion will allow us to formalize a major concern of policy makers and business leaders in developed countries, namely the fact that China and other developing countries are "moving up the value chain"; see e.g. OECD (2007).

Definition 3 *Let* $(w'_1, ..., w'_C)$ *denote the world income distribution in a counterfactual free trade equilibrium. Inequality is increasing (resp. decreasing) among a given group* $\{c_1, ...c_n\}$ *of adjacent countries if* $w'_{c+1}/w'_c \geq w_{c+1}/w_c$ *(resp.* $w'_{c+1}/w'_c \leq w_{c+1}/w_c$*) for all* $c_1 \leq c \leq c_n$*.*

Formally, inequality is rising or decreasing within a given group of adjacent countries if we can rank the vector of wages in the initial and counterfactual free trade equilibria in terms of the monotone likelihood ratio property. Since wages correspond to GDP per capita in our model, this property offers a simple way to conceptualize changes in the world income distribution in an economy with multiple countries.

4.2 Population Growth

In this section we study the impact of an increase in the total endowment of labor, L_{c_0}, of a given country c_0, which we refer to as "population growth." We proceed in two steps. First, we use Lemma 2 to characterize the changes in the pattern of vertical specialization. Second, we turn to Lemma 3 to characterize the associated changes in the world income distribution. Our first comparative static result can be stated as follows.

[9] Of course, a given country may be neither moving up nor down after a change.

Proposition 2 *Population growth in country c_0 increases the measure of stages performed in this country and decreases the measure of stages performed in any other country. In turn, all countries $c < c_0$ move down the supply chain and all countries $c > c_0$ move up.*

The broad intuition behind Proposition 2 is simple. An increase in the supply of labor in one country tends to raise total output at all stages of production. Since labor supply must remain equal to labor demand, this increase in output levels must be accompanied by a decrease in the measure of stages N_c performed in country $c \neq c_0$. Proceeding by iteration from the bottom and the top of the supply chain, we can then show that this change in N_c can only occur if all countries below c_0 move down and all countries above c_0 move up. Finally, since the total measure of stages must remain constant, the measure of stages N_{c_0} performed in country c_0 must increase.

Let us now use the previous result to analyze the consequences of population growth on the world income distribution.

Proposition 3 *Population growth in country c_0 decreases inequality among countries $c \in \{1, ..., c_0 - 1\}$, increases inequality among countries $c \in \{c_0, ..., c_1 - 1\}$, and decreases inequality among countries $c \in \{c_1, ..., C - 1\}$, with $c_1 \in \{c_0 + 1, ..., C\}$.*

The logic of Proposition 3 can be sketched as follows. From Lemma 3, we know that relative wages satisfy

$$\frac{w_{c+1}}{w_c} = 1 + \frac{\lambda_c - \lambda_{c+1}}{(w_c/p_c)}, \text{ for all } c < C. \tag{9}$$

Thus, w_{c+1}/w_c is decreasing in the labor intensity, w_c/p_c, of country c's export. From Proposition 2, we also know that countries at the bottom of the chain: (i) are moving down into lower stages, which tend to have lower export prices; and (ii) are performing fewer stages, which tends to reduce export prices (for a given price of imported inputs). Both effects tend to reduce the price of intermediate goods traded in that region of the supply chain, and in turn, to increase their labor intensity. This explains why inequality between nations declines at the bottom of the chain. The non-monotonic effects on inequality at the top of the chain reflect two conflicting forces. On the one hand, countries are moving up, which tends to increase the price of intermediate goods traded in that region of the supply chain, and in turn, to decrease their labor intensity. On the other hand, countries are performing fewer stages, which tends to reduce the price of their exports and increase their labor intensity.[10]

[10] Note that since $c_1 \in \{c_0 + 1, ..., C\}$, the third group of countries, $\{c_1, ..., C - 1\}$, is non-empty if $c_1 < C$, but empty if $c_1 = C$. In our simulations, we have encountered both situations.

The core mechanism behind Proposition 2 is reminiscent of the mechanism underlying terms-of-trade effects in a standard Ricardian model; see e.g. Dornbusch, Fischer, and Samuelson (1977) and Krugman (1986). From an economic standpoint, equation (9) captures the basic idea that the wage of country $c+1$ should decrease relative to the wage of country c if and only if it moves into sectors in which it has a comparative disadvantage. In our model, since country $c+1$ has a higher wage, these are the sectors with higher labor intensities. In a standard Ricardian model, this would be the sectors in which country $c+1$ is relatively less productive instead.

There is, however, one important difference between our elementary theory of trade with sequential production and a standard Ricardian model. In our model, the pattern of comparative advantage depends on *endogenous* differences in labor intensity across stages. In a standard Ricardian model, the same pattern only depends on *exogenous* productivity differences. This subtle distinction explains why our simple model is able to predict non-monotonic changes in inequality among the richest countries, whereas a standard Ricardian model would not.[11]

4.3 TFP Growth

We now turn our attention to the consequences of a decrease in the failure rate λ_{c_0} of a given country c_0, which we refer to as "TFP growth." For simplicity we restrict ourselves to a small change in λ_{c_0}, in the sense that it does not affect the ranking of countries in terms of failure rates. As in the previous section, we start by describing changes in the pattern of vertical specialization.

Proposition 4 *TFP growth in country c_0 increases the measure of stages performed in all countries $c < c_0$ and decreases the measure of stages performed in all countries $c > c_0$. In turn, all countries $c \neq c_0$ move up the supply chain.*

According to Proposition 4, all countries move up the supply chain in response to productivity growth in one country. Note that the consequences of TFP growth in country c_0 are the same as the consequences of population growth at the top of the chain, but the exact

[11] To see this, consider the closed economy model of Costinot and Vogel (2010). By relabelling "workers" as countries, this model can be reinterpreted as a standard Ricardian model with a large number of countries and goods, but no sequential production. The pattern of international specialization in this model also consists of a ladder of countries, with "high skill" countries specializing in the "high skill" goods. In this environment, unlike in the present model, population growth in one country can decrease inequality only among low skill countries and increase inequality only among high skill countries; see pages 758-759.

opposite at the bottom. To understand this result, consider first countries located at the top of the chain. Since total output of the final good must rise in response to an increase in productivity in country c_0, countries at the top of the chain must perform fewer stages for labor markets to clear. By a simple iterative argument, these countries must therefore move further up the supply chain, just like in Proposition 2.

The logic behind our results at the bottom of the chain is more subtle. The broad intuition behind the opposite effects of population and TFP growth can be understood as follows. Holding the pattern of vertical specialization fixed, population growth in country c_0 increases the total labor supply of countries $c \geq c_0$, but leaves their labor demand unchanged. Thus labor market clearing requires countries at the bottom of the chain to reduce the number of stages they perform, to move down the chain, and to increase their output, thereby offsetting the excess labor supply at the top. By contrast, TFP growth in country c_0 increases the total labor demand of countries $c \geq c_0$ (since country c_0 now produces more output at each stage), but leaves their labor supply unchanged. As a result, countries at the bottom of the chain now need to increase the number of stages they perform, to move up the chain, and to reduce their output in order to offset the excess labor demand at the top.[12]

In sharp contrast to a standard Ricardian model without sequential production, our model therefore predicts that population and TFP growth may have opposite effects on the pattern of international specialization. Building solely on the idea that labor markets must clear both before and after a given TFP shock, Proposition 4 illustrates how, through the fragmentation of the production process across borders, productivity growth in one country may lead all its trading partners to move up the supply chain, even in the absence of TFP growth in any of these countries.

The previous results have important implications for the world income distribution, which we now describe.

Proposition 5 *TFP growth in country c_0 increases inequality among countries $c \in \{1, ..., c_0 - 1\}$, decreases inequality among countries $c \in \{c_0, ..., c_1 - 1\}$, increases inequality among countries $c \in \{c_1, ..., c_2 - 1\}$, and decreases inequality among countries $c \in \{c_2, ..., C - 1\}$, with $c_1 \leq c_2$ and $c_1, c_2 \in \{c_0, ..., C\}$.*

[12] It is worth pointing out that while technological change in c_0 has unambiguous effects on N_c for all $c \neq c_0$, the effect on N_{c_0} is ambiguous. According to Proposition 4, technological change in the country at the bottom (top) of the chain necessarily increases (decreases) the measure of stages it performs. Thus N_{c_0} may either increase or decrease.

Like Section 4.2, changes in the pattern of vertical specialization naturally translate into changes in inequality between countries. At the top of the chain, the consequences of TFP growth for inequality are the same as the consequences of population growth, echoing the results of Propositions 2 and 4. The non-monotonicity—with inequality rising among countries $c \in \{c_1, ..., c_2 - 1\}$ and decreasing among countries $c \in \{c_2, ..., C - 1\}$—arises from the same two conflicting forces: countries move up the chain but produce fewer stages.

At the bottom and in the middle of the chain, by contrast, the consequences of TFP growth for inequality are different from those of population growth. In the middle of the chain, inequality may decrease among countries $c \in \{c_0, ..., c_1 - 1\}$ because of the direct effect of a reduction in λ_{c_0}, which tends to decrease inequality between c_0 and $c_0 + 1$, as seen in equation (9). This force was absent from our previous comparative static exercise since labor endowments did not directly affect zero profit conditions. At the bottom of the chain, $c < c_0$, our simple model again leads to sharp predictions regarding the implications of structural changes on inequality between countries. The logic is the same as in the previous section. From Proposition 4, we know that low TFP countries are both moving up the supply chain and performing more stages. Both effects tend to increase the price of the intermediate goods that they trade, and in turn, to decrease their labor intensity. Since richer countries are those with a comparative advantage in these goods, TFP growth in one country raises inequality between nations at the bottom of the chain. Though poor countries are moving up the supply chain—an observation which is often taken as "good news" among policy makers—they end up relatively worse off.

5 Extensions

Our elementary theory of trade with sequential production is special along several dimensions. A natural concern is the robustness of our main results to various modifications of some of our key assumptions. To address this issue we next present a number of extensions of our basic framework that incorporate more realistic features of global supply chains.

5.1 Sequential and Simultaneous Production

Most production processes are neither purely sequential, as assumed in this paper, nor purely simultaneous, as assumed in most of the existing literature. Producing an aircraft, for example, requires multiple parts. Many of these parts, e.g., engine, seats, windows, are produced simultaneously before being assembled, but each of these parts also require a large number of sequential stages, e.g. extraction of raw materials, refining, and manufacturing.[13] With that in mind, we turn to a generalization of our original model that allows for the coexistence of multiple supply chains, indexed by $n \in \mathcal{N} \equiv \{1, ..., N\}$. Beside its added realism, one benefit of this extension is that it also provides a strict generalization of a standard Ricardian model model with simultaneous production.

Each supply chain is as described in Section 2, except for the fact that the amount of labor L_c^n, allocated to chain n in country c is now endogenous. We denote by $\lambda_c^n > 0$ the exogenous rate at which country c makes mistakes if participating in supply chain n.

[**TO BE COMPLETED**]

5.2 Heterogeneous Stages of Production

In order to focus attention in the simplest possible way on the novel aspects of an environment with sequential production, we have assumed that stages of production only differ in one dimension: the order in which they are performed. In practice, stages of production often also differ greatly in terms of factor intensity, with some stages being much more skill-intensive than others. To capture such considerations within our simple theoretical framework, we now allow failure rates to be an exogenous characteristic of both a stage and a country. Among other things, this extension will give us a simple way to rationalize why assembly, a low-skill part of the production process, tends to occur in poor countries in practice.

Formally, we assume that mistakes occur at a Poisson rate, $\lambda_c(s) \equiv \theta(s) \lambda_c$, with $\theta(s)$ the "skill-intensity" of stage s. We come back to this terminology in a moment.

[**TO BE COMPLETED**]

5.3 Trade Costs

An important insight of the recent trade literature is that changes in trade costs affect the pattern and consequences of international trade not only by affecting final goods trade, but also by affecting the extent of production fragmentation across borders; see e.g. Feenstra and Hanson (1996), Yi (2003), and Grossman and Rossi-Hansberg (2008). We now discuss how the introduction of iceberg trade costs in our simple environment would affect the geographic structure of global supply chains, and in turn, the interdependence of nations. [**TO BE COMPLETED**]

[13] Of course, "manufacturing" itself requires a large number of sequential stages.

6 Concluding Remarks

In this paper, we have developed an elementary theory of trade with sequential production. In spite of its extreme simplicity, the predictions of our model are consistent with a number of stylized facts: (*i*) poor countries have higher shares of primary production in value added; (*ii*) rich countries tend to trade relatively more with other rich countries, while poor countries tend to trade relatively more with other poor countries; and (*iii*) rich countries both import and export goods with higher unit values. In our theory, none of these predictions rely on non-homothetic preferences, as commonly assumed in the existing literature. Instead they capture the basic idea that countries with different levels of GDP per capita tend to operate in different regions of the same supply chain.

Using this elementary theory, we have taken a first step towards analyzing how vertical specialization shapes the interdependence of nations. Among other things, we have shown that TFP growth in any country participating in a global supply chain leads all its trading partners to "move up" the chain, and in turn, always increases inequality between countries at the bottom of the chain. These results point towards the importance of modelling the sequential nature of production for understanding the consequences of technological changes in developing and developed countries on their trading partners worldwide.

Finally, while we have emphasized the consequences of vertical specialization for the interdependence of nations, we believe that our general results also have useful applications outside of international trade. Sequential production process are pervasive in practice. They may involve workers of different skills, as emphasized in the labor and organizations literature. They may also involve firms of different productivity, as in the industrial organization literature. Whatever the particular context may be, our theoretical analysis may help shed a new light on how vertical specialization shapes the interdependence between different actors of a given supply chain.

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A Proofs (I): Free Trade Equilibrium

Proof of Lemma 1. We proceed in four steps.

Step 1: $p(\cdot)$ *is continuous.*

Consider a stage $s_0 \in (0,1]$. By equations (3) and (4), we know that there must be at least one country, call it c_0, producing intermediate good s_0. By assumption, this country must also be producing all intermediate goods $s \in (s_0 - ds, s_0]$. Thus condition (2) implies

$$p(s_0) = (1 + \lambda_{c_0} ds) p(s_0 - ds) + w_{c_0} ds,$$

Taking the limit of the previous expression as ds goes to zero, we get

$$p(s_0) = \lim_{s \to s_0^-} p(s). \tag{10}$$

This already establishes the continuity of $p(\cdot)$ at 1. Now consider $s_0 \in (0,1)$. By the same logic, there must be at least one country, call it c_0 again, producing all intermediate goods $s \in (s_0, s_0 + ds]$. Condition (2) therefore also implies

$$p(s_0 + ds) = (1 + \lambda_{c_0} ds) p(s_0) + w_{c_0} ds,$$

Taking the limit of the previous expression as ds goes to zero, we then get

$$\lim_{s \to s_0^+} p(s) = p(s_0). \tag{11}$$

The continuity of $p(\cdot)$ at $s_0 \in (0,1)$ directly derives from equations (10) and (11).

Step 2: $p(\cdot)$ *is strictly increasing.*

We proceed by contradiction. Suppose that there exists a pair of stages, s_1 and s_2, such that $s_1 < s_2$ and $p(s_1) \geq p(s_2)$. Since $p(\cdot)$ is continuous, there must also exist a stage $s_0 \in (s_1, s_2]$ and an $\varepsilon > 0$ such that $p(s) \geq p(s_0)$ for all $s \in (s_0 - \varepsilon, s_0]$. As in Step 1, we know that there must be at least one country, call it c_0, producing intermediate good s_0. This, in turn, requires

$$p(s_0) = (1 + \lambda_{c_0} ds) p(s_0 - ds) + w_{c_0} ds > p(s_0 - ds).$$

For ds small enough, the previous inequality contradicts $p(s) \geq p(s_0)$ for all $s \in (s_0 - \varepsilon, s_0]$.

Step 3: *If* $c_2 > c_1$*, then* $w_{c_2} > w_{c_1}$*.*

We proceed by contradiction. Suppose that there exist two countries, $c_2 > c_1$, such that $w_{c_2} \leq w_{c_1}$. In a free trade equilibrium, equations (3) and (4) require country c_1 to produce at least one intermediate good in $(0,1)$, call it s_1. By equation (1), this country must also be producing all intermediate goods $s \in (s_1 - ds, s_1]$. Thus condition (2) implies

$$
\begin{aligned}
p(s_1) &= (1 + \lambda_{c_1} ds)\, p(s_1 - ds) + w_{c_1} ds, & (12)\\
p(s_1) &\leq (1 + \lambda_{c_2} ds)\, p(s_1 - ds) + w_{c_2} ds, & (13)
\end{aligned}
$$

Since $\lambda_{c_2} < \lambda_{c_1}$, equation (12) and inequality (13) imply $w_{c_2} > w_{c_1}$, which contradicts $w_{c_2} \leq w_{c_1}$.

Step 4: *If $c_2 > c_1$ and $Q_{c_1}(s_1) > 0$, then $Q_{c_2}(s) = 0$ for all $s < s_1$.*

We proceed by contradiction. Suppose that there exist two countries, $c_2 > c_1$, and two intermediate goods, $s_1 > s_2 > 0$, such that c_1 produces s_1 and c_2 produces s_2. By equation (1), c_1 produces all intermediate goods $s \in (s_1 - ds, s_1]$, whereas c_2 produces all intermediate goods $s \in (s_2 - ds, s_2]$. Thus condition (2) implies

$$
\begin{aligned}
p(s_1) &= (1 + \lambda_{c_1} ds)\, p(s_1 - ds) + w_{c_1} ds,\\
p(s_2) &= (1 + \lambda_{c_2} ds)\, p(s_2 - ds) + w_{c_2} ds,\\
p(s_1) &\leq (1 + \lambda_{c_2} ds)\, p(s_1 - ds) + w_{c_2} ds,\\
p(s_2) &\leq (1 + \lambda_{c_1} ds)\, p(s_2 - ds) + w_{c_1} ds.
\end{aligned}
$$

Combining the four previous expressions, we get

$$
\begin{aligned}
& [(1 + \lambda_{c_2} ds)\, p(s_1 - ds) + w_{c_2} ds]\, [(1 + \lambda_{c_1} ds)\, p(s_2 - ds) + w_{c_1} ds]\\
\geq\; & [(1 + \lambda_{c_1} ds)\, p(s_1 - ds) + w_{c_1} ds]\, [(1 + \lambda_{c_2} ds)\, p(s_2 - ds) + w_{c_2} ds],
\end{aligned}
$$

which can be rearranged as

$$
\begin{aligned}
& (1 + \lambda_{c_2} ds)\, [p(s_1 - ds) - p(s_2 - ds)]\, w_{c_1}\\
\geq\; & (1 + \lambda_{c_1} ds)\, [p(s_1 - ds) - p(s_2 - ds)]\, w_{c_2}
\end{aligned}
$$

By Step 2, we know that $p(s_1 - ds) - p(s_2 - ds) > 0$. Thus the previous inequality implies

$$
(1 + \lambda_{c_2} ds)\, w_{c_1} \geq (1 + \lambda_{c_1} ds)\, w_{c_2}. \tag{14}
$$

Since $\lambda_{c_2} < \lambda_{c_1}$, inequality (14) implies $w_{c_1} > w_{c_2}$, which contradicts Step 3.

To conclude the proof of Lemma 1, let us define $S_c \equiv \sup\{s \in \mathcal{S} | Q_c(s) > 0\}$ for all $c \in \mathcal{C}$. By equation Step 4, we must have $S_0 \equiv 0 < S_1 < ... < S_C = 1$, and for all $s \in \mathcal{S}$ and $c \in \mathcal{C}$, $Q_c(s) > 0$ if $S_{c-1} < s < S_c$ and $Q_c(s) = 0$ if $s < S_{c-1}$ or $s > S_c$. Since $Q_c(s) > 0$ requires $Q_c(s') > 0$ for all $s' \in (s - ds, s]$, we must also have $Q_c(S_c) > 0$ and $Q_c(S_{c-1}) = 0$ for all $c \in \mathcal{C}$. Thus $Q_c(s) > 0$ if and only if $s \in (S_{c-1}, S_c]$. Finally, by equations (3) and (4), country C must produce stage 1, so that $S_C = 1$. **QED**. ■

Proof of Lemma 2. We first consider equation (6). Lemma 1 and equation (3) imply

$$Q_c(s_2) - Q_c(s_1) = -\lambda_c \int_{s_1}^{s_2} Q_c(s)\, ds, \text{ for all } s_1, s_2 \in (S_{c-1}, S_c]. \tag{15}$$

Taking the derivative of the previous expression with respect to s_2, we get

$$dQ_c(s)/ds = -\lambda_c Q_c(s), \text{ for all } s \in (S_{c-1}, S_c].$$

The solution of the previous differential equation must satisfy

$$Q_c(S_c) = e^{-\lambda_c(S_c - S_{c-1})} \lim_{s \to S_{c-1}^+} Q_c(s). \tag{16}$$

Lemma 1 and equation (3) also imply

$$Q_c(S_{c-1} + ds) - Q_{c-1}(S_{c-1} - ds) = -\left[\lambda_c \lim_{s \to S_{c-1}^+} Q_c(s) + \lambda_{c-1} Q_{c-1}(S_{c-1} - ds)\right] ds.$$

Taking the limit of the previous expression as ds goes to zero, we get

$$\lim_{s \to S_{c-1}^+} Q_c(s) = \lim_{s \to S_{c-1}^-} Q_{c-1}(s) = Q_{c-1}(S_{c-1}). \tag{17}$$

Equation (6) derives from equations (16) and (17) and the definition of $Q_c \equiv Q_c(S_c)$.

Let us now turn to equation (5). By Lemma 1 and equation (4), we know that

$$\int_{S_{c-1}}^{S_c} Q_c(s)\, ds = L_c, \text{ for all } c \in \mathcal{C}. \tag{18}$$

By equations (15) and (17), we also know that

$$\int_{S_{c-1}}^{S_c} Q_c(s)\, ds = \frac{1}{\lambda_c}\left[Q_{c-1}(S_{c-1}) - Q_c(S_c)\right]. \tag{19}$$

Equations (18) and (19) imply

$$L_c = \frac{1}{\lambda_c}\left[Q_{c-1}(S_{c-1}) - Q_c(S_c)\right], \text{ for all } c \in \mathcal{C}. \tag{20}$$

Equation (5) derives from equations (6) and (20) and the definition of $Q_c \equiv Q_c(S_c)$. The boundary conditions $S_0 = 0$ and $S_C = 1$ have already been established in the proof of Lemma 1. **QED.** ■

Proof of Lemma 3. We first consider equation (7). Lemma 1 and condition (2) imply

$$\begin{aligned} p\left(S_c+ds\right)-\left(1+\lambda_{c+1}ds\right)p\left(S_c\right)-w_{c+1}ds &\geq p\left(S_c+ds\right)-\left(1+\lambda_c ds\right)p\left(S_c\right)-w_c ds, \\ p\left(S_c\right)-\left(1+\lambda_c ds\right)p\left(S_c-ds\right)-w_c ds &\geq p\left(S_c\right)-\left(1+\lambda_{c+1}ds\right)p\left(S_c-ds\right)-w_{c+1}ds. \end{aligned}$$

After simplifications, the two previous inequalities can be rearranged as

$$\left(\lambda_c-\lambda_{c+1}\right)p\left(S_c\right)\geq w_{c+1}-w_c\geq\left(\lambda_c-\lambda_{c+1}\right)p\left(S_c-ds\right).$$

Since p is continuous, taking the limit of the above chain of inequalities as ds goes to zero we get

$$w_{c+1}-w_c=\left(\lambda_c-\lambda_{c+1}\right)p\left(S_c\right),\text{ for all } s\in(S_{c-1},S_c].$$

which is equivalent to equation (7) by the definition of $p_c\equiv p\left(S_c\right)$.

Let us now turn to equation (8). Lemma 1 and condition (2) imply

$$p\left(s+ds\right)=\left(1+\lambda_c ds\right)p\left(s\right)+w_c ds$$

Taking the limit of the previous expression as ds goes to zero, we get

$$dp\left(s\right)/ds=\lambda_c p\left(s\right)+w_c,\text{ for all } s\in(S_{c-1},S_c].$$

The solution of the previous differential equation must satisfy

$$p\left(S_c\right)=e^{\lambda_c(S_c-S_{c-1})}\lim_{s\to S_{c-1}^+}p\left(S_{c-1}\right)+\left[\frac{e^{\lambda_c(S_c-S_{c-1})}-1}{\lambda_c}\right]\left(w_c/\lambda_c\right),$$

which is equivalent to equation (8) by the continuity of $p\left(\cdot\right)$ and the definitions of $N_c\equiv S_c-S_{c-1}$ and $p_c\equiv p\left(S_c\right)$. The boundary conditions derive from the fact that $p_0=p\left(S_0\right)=p(0)=0$ and $p_C=p\left(S_C\right)=p\left(1\right)=1$. **QED.** ∎

Proof of Proposition 1. We proceed in four steps.

Step 1: $(S_0,...,S_C)$ *and* $(Q_0,...,Q_C)$ *satisfy equations* (5) *and* (6) *if and only if*

$$\begin{aligned} S_c &= S_0+\textstyle\sum_{c'=1}^{c}\left(\dfrac{1}{\lambda_{c'}}\right)\ln\left[\dfrac{Q_0-\sum_{c''=1}^{c'-1}\lambda_{c''}L_{c''}}{Q_0-\sum_{c''=1}^{c'}\lambda_{c''}L_{c''}}\right],\ \textit{for all } c\in\mathcal{C}, &&(21)\\ Q_c &= Q_0-\textstyle\sum_{c'=1}^{c}\lambda_{c'}L_{c'},\ \textit{for all } c\in\mathcal{C}. &&(22) \end{aligned}$$

Let us first show that if $(S_0,...,S_C)$ and $(Q_0,...,Q_C)$ satisfy equations (5) and (6), then they satisfy equations (21) and (22). Consider equation (22). Equations (5) and (6) imply

$$Q_c=Q_{c-1}-\lambda_c L_c,\text{ for all } c\in\mathcal{C},$$

By iteration we therefore have

$$Q_c = Q_0 - \sum_{c'=1}^{c} \lambda_{c'} L_{c'}, \text{ for all } c \in \mathcal{C}.$$

Now consider equation (21). Starting from equation (5) and iterating we get

$$S_c = S_0 - \sum_{c'=1}^{c} \left(\frac{1}{\lambda_{c'}}\right) \ln\left(1 - \frac{\lambda_{c'} L_{c'}}{Q_{c'-1}}\right), \text{ for all } c \in \mathcal{C}.$$

Equation (21) directly derives from the previous expression and equation (22). It is a matter of simple algebra to check that if $(S_0, ..., S_C)$ and $(Q_0, ..., Q_C)$ satisfy equations (21) and (22), then they satisfy equations (5) and (6).

Step 2: *There exists a unique pair of vectors* $(S_0, ..., S_C)$ *and* $(Q_0, ..., Q_C)$ *satisfying equations* (5) *and* (6) *and the boundary conditions:* $S_0 = 0$ *and* $S_C = 1$.

Let $\underline{Q}_0 \equiv \sum_{c=1}^{C} \lambda_c L_c$. By Step 1, if $Q_0 \leq \underline{Q}_0$, then there does not exist a pair of vectors $(S_0, ..., S_C)$ and $(Q_0, ..., Q_C)$ that satisfy equations (5) and (6). Otherwise $(Q_0, ..., Q_C)$ and $(S_0, ..., S_C)$ would also satisfy equations (21) and (22), which cannot be the case if $Q_0 \leq \underline{Q}_0$. Now consider $Q_0 > \underline{Q}_0$. From equation (21), it is easy to check that $\partial S_C / \partial Q_0 < 0$ for all $Q_0 > \underline{Q}_0$; $\lim_{Q_0 \to \underline{Q}_0^+} S_C = +\infty$; and $\lim_{Q_0 \to +\infty} S_C = S_0$. Thus conditional on having set $S_0 = 0$, there exists a unique $Q_0 > \underline{Q}_0$ such that $(S_0, ..., S_C)$ and $(Q_0, ..., Q_C)$ satisfy equations (21) and (22) and $S_C = S$. Step 2 derives from Step 1 and the previous observation.

Step 3: *For any* $(N_1, ..., N_C)$*, there exists a unique pair of vectors* $(w_1, ..., w_C)$ *and* $(p_0, ..., p_C)$ *satisfying equations* (7) *and* (8) *and the boundary conditions:* $p_0 = 0$ *and* $p_C = 1$.

For any $(N_1, ..., N_C)$, w_1, and p_0, there trivially exists a unique pair of vectors $(w_2, ..., w_C)$ and $(p_1, ..., p_C)$ that satisfy equations (7) and (8). Thus taking $(N_1, ..., N_C)$ as given and having set $p_0 = 0$, we only need to check that there exists a unique w_1 such that $p_C = 1$. To do so, we first establish that p_C is strictly increasing in w_1. We proceed by iteration. By equation (8), we know that p_1 is strictly increasing in w_1. Thus by equation (7), w_2 must be strictly increasing in w_1 as well. Now suppose that p_{c-1} and w_c are strictly increasing in w_1 for $c < C$. Then p_c must be strictly increasing in w_1, by equation (8), w_{c+1} must be strictly increasing in w_1, by equation (7). At this point we have established, by iteration, that p_{C-1} and w_C are strictly increasing in w_1. Combining this observation with equation (8), we obtain that p_C is strictly increasing in w_1. To conclude, let us note that, by equations (7) and (8), we also have $\lim_{w_1 \to 0} p_C = 0$ and $\lim_{w_1 \to +\infty} p_C = +\infty$. Since p_C is strictly increasing in w_1, there therefore exists a unique w_1 such that $p_C = 1$.

Steps 1-3 imply the existence and uniqueness of $(S_0, ..., S_C)$, $(Q_0, ..., Q_C)$, $(w_1, ..., w_C)$, and $(p_0, ..., p_C)$ that satisfy equations (5)-(8) with boundary conditions $S_0 = 0$, $S_C = 1$, $p_0 = 0$, and $p_C = 1$. Now consider the following output levels and intermediate good prices

$$\begin{aligned} Q_c(s) &= e^{-\lambda_c (s - S_{c-1})} Q_{c-1}, \text{ for all } s \in (S_{c-1}, S_c], \\ p(s) &= e^{\lambda_c (s - S_{c-1})} p_{c-1} + \left[e^{\lambda_c (s - S_{c-1})} - 1\right] (w_c / \lambda_c) \text{ for all } s \in (S_{c-1}, S_c]. \end{aligned}$$

By construction, $[Q_1(\cdot), ..., Q_C(\cdot)]$, $(w_1, ..., w_C)$, and $p(\cdot)$ satisfy conditions (2)-(4). Thus a free trade equilibrium exists. Since $(S_0, ..., S_C)$, $(Q_0, ..., Q_C)$, $(w_1, ..., w_C)$, and $(p_0, ..., p_C)$ are unique, the free trade equilibrium is unique as well by Lemmas 1-3. **QED.** ■

B Proofs (II): Population Growth

Proof of Proposition 2. Throughout this proof, we use the two following relationships

$$\begin{aligned} Q_{c-1} &= \frac{\lambda_c L_c}{1 - e^{-\lambda_c N_c}}, \text{ for all } c \in \mathcal{C} &(23)\\ Q_c &= \frac{\lambda_c L_c e^{-\lambda_c N_c}}{1 - e^{-\lambda_c N_c}}, \text{ for all } c \in \mathcal{C}, &(24)\end{aligned}$$

Equation (23) derives from equation (5) and the definition of $N_c \equiv S_c - S_{c-1}$. Equation further uses equation (6). For future reference, note that $\partial Q_{c-1}/\partial N_c < 0$ and $\partial Q_c/\partial N_c < 0$, whereas $\partial Q_{c-1}/\partial L_c > 0$ and $\partial Q_c/\partial L_c > 0$. The rest of our proof proceeds in six steps.

Step 1: *If $L'_{c_0} > L_{c_0}$ and $c_0 \neq C$, then $N'_C < N_C$.*

We first establish that $Q'_C > Q_C$. By Lemma 1, we know that $Q'_C \equiv Q'_C(1) = \sum_{c=1}^{C} Q'_c(1)$. By the First Welfare Theorem, we also know that the allocation in a free trade equilibrium is Pareto optimal. Thus Q'_C must be the maximum output level of the final good attainable given the new resource and technological constraints, i.e.,

$$Q'_C = \underset{\widetilde{Q}_1(\cdot),...,\widetilde{Q}_C(\cdot)}{\arg\max} \ \textstyle\sum_{c=1}^{C} \widetilde{Q}_c(1),$$

subject to

$$\begin{aligned} \textstyle\sum_{c=1}^{C} \widetilde{Q}_c(s_2) - \sum_{c=1}^{C} \widetilde{Q}_c(s_1) &\leq -\textstyle\int_{s_1}^{s_2} \sum_{c=1}^{C} \lambda_c \widetilde{Q}_c(s)\, ds, \text{ for all } s_1 \leq s_2, &(25)\\ \int_0^S \widetilde{Q}_c(s)\, ds &\leq L'_c, \text{ for all } c \in \mathcal{C}, &(26)\end{aligned}$$

where $L'_{c_0} > L_{c_0}$ and $L'_c = L_c$ for all $c \neq c_0$. Now consider $\widetilde{Q}_1(\cdot), ..., \widetilde{Q}_C(\cdot)$ such that

$$\widetilde{Q}_{c_0}(s) \equiv Q_{c_0}(s) + \left(\frac{\lambda_{c_0} e^{-\lambda_{c_0} s}}{1 - e^{-\lambda_{c_0} S}}\right)\left(L'_{c_0} - L_{c_0}\right), \text{ for all } s \in \mathcal{S},$$

and

$$\widetilde{Q}_c(s) \equiv Q_c(s), \text{ for all } s \in \mathcal{S} \text{ and } c \neq c_0.$$

Since $Q_1(\cdot), ..., Q_C(\cdot)$ satisfies the initial resource and technological constraints, as described by conditions (3) and (4), $\widetilde{Q}_1(\cdot), ..., \widetilde{Q}_C(\cdot)$ must satisfy, by construction, the new resource and technological constraints, as described by conditions (25) and (26). Since $L'_{c_0} > L_{c_0}$, we must also have

$$\widetilde{Q}_{c_0}(1) + \widetilde{Q}_C(1) = \left(\frac{\lambda_{c_0} e^{-\lambda_{c_0} S}}{1 - e^{-\lambda_{c_0} S}}\right)\left(L'_{c_0} - L_{c_0}\right) + Q_C > Q_C.$$

Since $Q'_C \geq \widetilde{Q}_{c_0}(1) + \widetilde{Q}_C(1)$, the previous inequality implies $Q'_C > Q_C$. Combining this observation with equation (24) and the fact that $c_0 \neq C$, which implies $L'_C = L_C$, we get $N'_C < N_C$.

Step 2: *If $L'_{c_0} > L_{c_0}$, then $N'_c < N_c$ for all $c > c_0$.*

We proceed by iteration. By Step 1, we know that if $L'_{c_0} > L_{c_0}$ and $c_0 \neq C$, then $N'_C < N_C$. Let us now show that if $N'_c < N_c$ and $c > c_0 + 1$, then $N'_{c-1} < N_{c-1}$. Since $c > c_0 + 1$, we know that $L'_c = L_c$. Thus $N'_c < N_c$ and equation (23) imply $Q'_{c-1} > Q_{c-1}$. Since $c > c_0 + 1$, we also know that $L'_{c-1} = L_{c-1}$. Thus $Q'_{c-1} > Q_{c-1}$ and equation (24) imply $N'_{c-1} < N_{c-1}$. This completes the proof of Step 2.

Step 3: *If $L'_{c_0} > L_{c_0}$ and $c_0 \neq 1$, then $N'_1 < N_1$.*

We proceed by contradiction. Suppose that $N'_1 \geq N_1$. Since $c_0 \neq 1$, we know that $L'_1 = L_1$. Thus $N'_1 \geq N_1$ and equation (24) imply $Q'_1 \leq Q_1$. Let us now show that if $N'_c \geq N_c$ and $Q'_c \leq Q_c$ for $c < c_0 - 1$, then $N'_{c+1} \geq N_{c+1}$ and $Q'_{c+1} \leq Q_{c+1}$. Since $c < c_0 - 1$, we know that $L'_{c+1} = L_{c+1}$. Thus $Q'_c \leq Q_c$ and equation (23) imply $N'_{c+1} \geq N_{c+1}$, whereas $N'_{c+1} \geq N_{c+1}$ and equation (24) imply $Q'_{c+1} \leq Q_{c+1}$. At this point, we have established, by iteration, that $N'_c \geq N_c$ and $Q'_c \leq Q_c$ for all countries $c < c_0$. Let us now turn to countries $c \geq c_0$. First consider country c_0. By assumption, we know that $L'_{c_0} > L_{c_0}$. Thus $Q'_{c_0-1} \leq Q_{c_0-1}$ and equation (23) imply $N'_{c_0} > N_{c_0}$, whereas $N'_{c_0} \geq N_{c_0}$ and equation (24) imply $Q'_{c_0} < Q_{c_0}$. Using the same logic as before, let us now show that if $N'_c > N_c$ and $Q'_c < Q_c$ for $c \geq c_0$, then $N'_{c+1} > N_{c+1}$ and $Q'_{c+1} < Q_{c+1}$. Since $c \geq c_0$, we know that $L'_{c+1} = L_{c+1}$. Thus $Q'_c < Q_c$ and equation (23) imply $N'_{c+1} > N_{c+1}$, whereas $N'_{c+1} > N_{c+1}$ and equation (24) imply $Q'_{c+1} < Q_{c+1}$. At this point, we have therefore established, by iteration, that $N'_c \geq N_c$ and $Q'_c \leq Q_c$ for all countries $c \in \mathcal{C}$, with strict inequality for all countries $c \geq c_0$. This implies $\sum_{c=1}^{C} N'_c = S'_C - S'_0 > S_C - S_0 = \sum_{c=1}^{C} N_c$, which contradicts $S'_C - S'_0 = S_C - S_0 = 1$ by Lemma 2.

Step 4: *If $L'_{c_0} > L_{c_0}$, then $N'_c < N_c$ for all $c < c_0$.*

We proceed by iteration. By Step 3, we know that if $L'_{c_0} > L_{c_0}$ and $c_0 \neq 1$, then $N'_1 < N_1$. Since $c_0 \neq 1$, we know that $L'_1 = L_1$. Thus $N'_1 < N_1$ and equation (24) imply $Q'_1 > Q_1$. Let us now show that if $N'_c < N_c$ and $Q'_c > Q_c$ for $c < c_0 - 1$, then $N'_{c+1} < N_{c+1}$ and $Q'_{c+1} > Q_{c+1}$. Since $c < c_0 - 1$, we know that $L'_{c+1} = L_{c+1}$. Thus $Q'_c > Q_c$ and equation (23) imply $N'_{c+1} < N_{c+1}$, whereas $N'_{c+1} < N_{c+1}$ and equation (24) imply $Q'_{c+1} > Q_{c+1}$. This completes the proof of Step 4.

Step 5: *If $L'_{c_0} > L_{c_0}$, then $S'_c \geq S_c$ for all $c \geq c_0$.*

We proceed by iteration. By Lemma 2, we know that $S'_C = S_C = S$. Thus $S'_c \geq S_c$ is satisfied for $c = C$. Let us now show that if $S'_c \geq S_c$ and $c > c_0$, then $S'_{c-1} \geq S_{c-1}$. Since $c > c_0$, $L'_{c_0} > L_{c_0}$ and Step 2 imply $N'_c < N_c$. Combining this observation with $S'_c \geq S_c$ and the definition of $N_c \equiv S_c - S_{c-1}$, we obtain $S'_{c-1} \geq S_{c-1}$. This completes the proof of Step 5.

Step 6: *If $L'_{c_0} > L_{c_0}$, then $S'_{c-1} \leq S_{c-1}$ for all $c \leq c_0$.*

We proceed by iteration. By Lemma 2, we know that $S'_0 = S_0 = 0$. Thus $S'_{c-1} \leq S_{c-1}$ is satisfied for $c = 1$. Let us now show that if $S'_{c-1} \leq S_{c-1}$ and $c < c_0$, then $S'_c \leq S_c$. Since $c < c_0$, $L'_{c_0} > L_{c_0}$ and Step 4 imply $N'_c < N_c$. Combining this observation with $S'_{c-1} \leq S_{c-1}$ and the definition of $N_c \equiv S_c - S_{c-1}$, we obtain $S'_c \leq S_c$. This completes the proof of Step 6.

To conclude, note that Step 2, Step 4, and the fact that $\sum_{c=1}^{C} N'_c = \sum_{c=1}^{C} N_c = 1$, by Lemma 2, imply $N'_{c_0} > N_{c_0}$ and $N'_c < N_c$ for all $c \neq c_0$. Finally, Definition 2, Step 5, and Step 6 imply that all countries $c < c_0$ move down the supply chain and all countries $c > c_0$ move up. **QED.** ■

Proof of Proposition 3. We start by demonstrating the following Lemma.

Lemma 4 *For any country $1 < c < C$, if $N'_c < N_c$ and $(w_c/w_{c-1})' < (w_c/w_{c-1})$, then $(w_{c+1}/w_c)' < (w_{c+1}/w_c)$.*

Proof of Lemma 4. Consider a country $1 < c < C$. Equations (7) and (8) imply

$$\frac{w_{c+1}}{w_c} = 1 + (\lambda_c - \lambda_{c+1}) \left[\left(\frac{e^{\lambda_c N_c} - 1}{\lambda_c} \right) + e^{\lambda_c N_c} \left(\frac{w_{c-1}}{w_c} \right) \left(\frac{p_{c-1}}{w_{c-1}} \right) \right] \tag{27}$$

By equation (7), we also know that

$$\frac{w_c}{w_{c-1}} = 1 + (\lambda_{c-1} - \lambda_c) \left(\frac{p_{c-1}}{w_{c-1}} \right), \tag{28}$$

which further implies

$$\left(\frac{w_{c-1}}{w_c} \right) \left(\frac{p_{c-1}}{w_{c-1}} \right) = \frac{(p_{c-1}/w_{c-1})}{1 + (\lambda_{c-1} - \lambda_c)(p_{c-1}/w_{c-1})}. \tag{29}$$

Since $(w_c/w_{c-1})' < (w_c/w_{c-1})$ and $\lambda_{c-1} > \lambda_c$, equation (28) immediately implies

$$\left(\frac{p_{c-1}}{w_{c-1}} \right)' < \left(\frac{p_{c-1}}{w_{c-1}} \right).$$

Combining this observation with equation (29)—the right-hand side of which is increasing in (P_{c-1}/w_{c-1})—we obtain

$$\left(\frac{w_{c-1}}{w_c} \right)' \left(\frac{p_{c-1}}{w_{c-1}} \right)' < \left(\frac{w_{c-1}}{w_c} \right) \left(\frac{p_{c-1}}{w_{c-1}} \right). \tag{30}$$

To conclude, note that $N'_c < N_c$ implies $e^{\lambda_c N'_c} < e^{\lambda_c N_c}$. Thus equation (27) and inequality (30) imply $(w_{c+1}/w_c)' < (w_{c+1}/w_c)$. **QED.** ■

Proof of Proposition 3 (Continued). The rest of the proof of Proposition 3 proceeds in three steps.

Step 1: *If $L'_{c_0} > L_{c_0}$ and $c_0 \neq 1$, then $(w_2/w_1)' < (w_2/w_1)$.*

Equation (7), equation (8), and $p_0 = 0$ imply

$$\frac{w_2}{w_1} = 1 + \frac{1}{\lambda_1}(\lambda_1 - \lambda_2)\left(e^{\lambda_1 N_1} - 1\right). \tag{31}$$

By Proposition 2, we know that $L'_{c_0} > L_{c_0}$ implies $N'_1 < N_1$ if $c_0 \neq 1$. Combining this observation with equation (31), we obtain $(w_2/w_1)' < (w_2/w_1)$ if $c_0 \neq 1$.

Step 2: $(w_{c+1}/w_c)' < (w_{c+1}/w_c)$ *for all* $c < c_0$.

By Proposition 2, we know that $L'_{c_0} > L_{c_0}$ implies $N'_c < N_c$ for all $c < c_0$. Therefore, we can invoke Step 1 and Lemma 4 to establish, by induction, that $(w_{c+1}/w_c)' < (w_{c+1}/w_c)$ for all $c < c_0$.

Step 3: *If there exists* $\widetilde{c} \geq c_0$ *such that* $(w_{\widetilde{c}+1}/w_{\widetilde{c}})' < (w_{\widetilde{c}+1}/w_{\widetilde{c}})$, *then* $(w_{c+1}/w_c)' < (w_{c+1}/w_c)$ *for all* $\widetilde{c} \leq c < C$.

Suppose that there exists $\widetilde{c} \geq c_0$ such that $(w_{\widetilde{c}+1}/w_{\widetilde{c}})' < (w_{\widetilde{c}+1}/w_{\widetilde{c}})$. By Proposition 2, we know that $L'_{c_0} > L_{c_0}$ implies $N'_c < N_c$ for all $c > \widetilde{c} \geq c_0$. Therefore, we can again invoke Lemma 4 to establish, by induction, that $(w_{c+1}/w_c)' < (w_{c+1}/w_c)$ for all $\widetilde{c} \leq c < C$. This concludes the proof of Step 3. At this point, we can define $c_1 \equiv \inf\left\{c \geq c_0 | \, (w_{c+1}/w_c)' < (w_{c+1}/w_c)\right\}$. Step 2 implies that if $L'_{c_0} > L_{c_0}$, then w_{c+1}/w_c falls for all $c < c_0$, while Step 3 and the definition of c_1 implies that w_{c+1}/w_c rises for all $c_0 \leq c < c_1$ and falls for all $c_1 \leq c < C$. In order to establish Proposition 3, the only thing left to show is that $c_1 > c_0$, which is what we establish in our final step.

Step 4: *If* $L'_{c_0} > L_{c_0}$ *and* $c_0 \neq C$, *then* $(w_{c_0+1}/w_{c_0})' > (w_{c_0+1}/w_{c_0})$.

We proceed by contradiction. Suppose that $(w_{c_0+1}/w_{c_0})' \leq (w_{c_0+1}/w_{c_0})$. Then for any $\varepsilon > 0$, there must also exist $L_{c_0} \leq \tilde{L}_{c_0} < \tilde{L}'_{c_0} \leq L'_{c_0}$ such that $|\tilde{L}'_{c_0} - \tilde{L}_{c_0}| < \varepsilon$ and

$$(\tilde{w}_{c_0+1}/\tilde{w}_{c_0})' \leq (\tilde{w}_{c_0+1}/\tilde{w}_{c_0}), \tag{32}$$

where $(\tilde{w}_1, ..., \tilde{w}_C)$ and $(\tilde{w}'_1, ..., \tilde{w}'_C)$ denote the world income distribution if labor endowments in country c_0 are equal to $\tilde{L}_{c_0}$ and $\tilde{L}'_{c_0}$, respectively. For $|\tilde{L}'_{c_0} - \tilde{L}_{c_0}|$ small enough, Lemma 2 implies

$$\tilde{S}'_1 < \tilde{S}_1 < \tilde{S}'_2 < ... < \tilde{S}'_{c_0-1} < \tilde{S}_{c_0-1} < \tilde{S}_{c_0} < \tilde{S}'_{c_0} < \tilde{S}_{c_0+1} < ...., \tag{33}$$

where $\left(\tilde{S}_1, ..., \tilde{S}_C\right)$ and $\left(\tilde{S}'_1, ..., \tilde{S}'_C\right)$ denote the pattern of vertical specialization if labor endowments in country c_0 are equal to $\tilde{L}_{c_0}$ and $\tilde{L}'_{c_0}$, respectively. First, note that for any $c < c_0 - 1$, since $\tilde{S}'_c < \tilde{S}_c < \tilde{S}'_{c+1} < \tilde{S}_{c+1}$, condition (2) implies that

$$\begin{aligned}
\frac{\tilde{p}'\left(\tilde{S}'_{c+1}\right)}{\tilde{w}'_{c+1}} &= \frac{e^{\lambda_{c+1}\left(\tilde{S}'_{c+1}-\tilde{S}_c\right)}\tilde{p}'\left(\tilde{S}_c\right)}{\tilde{w}'_{c+1}} + \frac{e^{\lambda_{c+1}\left(\tilde{S}'_{c+1}-\tilde{S}_c\right)} - 1}{\lambda_{c+1}}, \\
\frac{\tilde{p}\left(\tilde{S}'_{c+1}\right)}{\tilde{w}_{c+1}} &= \frac{e^{\lambda_{c+1}\left(\tilde{S}'_{c+1}-\tilde{S}_c\right)}\tilde{p}\left(\tilde{S}_c\right)}{\tilde{w}_{c+1}} + \frac{e^{\lambda_{c+1}\left(\tilde{S}'_{c+1}-\tilde{S}_c\right)} - 1}{\lambda_{c+1}},
\end{aligned}$$

where $\tilde{p}(\cdot)$ and $\tilde{p}'(\cdot)$ denote the schedule of prices if labor endowments in country c_0 are equal to

$\widetilde{L}_{c_0}$ and $\tilde{L}'_{c_0}$, respectively. Since $\tilde{S}'_{c+1} > \tilde{S}_c$, the two previous equations further imply that for any $c < c_0 - 1$,

$$\tilde{p}'\left(\tilde{S}_c\right)/\tilde{w}'_{c+1} \geq \tilde{p}\left(\tilde{S}_c\right)/\tilde{w}_{c+1} \Rightarrow \tilde{p}'\left(\tilde{S}'_{c+1}\right)/\tilde{w}'_{c+1} \geq \tilde{p}\left(\tilde{S}'_{c+1}\right)/\tilde{w}_{c+1}. \tag{34}$$

Second, note that for any $c < c_0$, since $\tilde{S}_{c-1} < \tilde{S}'_c < \tilde{S}_c$, condition (2) also implies that

$$\frac{\tilde{p}'\left(\tilde{S}_c\right)}{\tilde{w}'_{c+1}} = \frac{e^{\lambda_{c+1}\left(\tilde{S}_c-\tilde{S}'_c\right)}\tilde{p}'\left(\tilde{S}'_c\right)}{\tilde{w}'_{c+1}} + \frac{e^{\lambda_{c+1}\left(\tilde{S}_c-\tilde{S}'_c\right)}-1}{\lambda_{c+1}}, \tag{35}$$

$$\frac{\tilde{p}\left(\tilde{S}_c\right)}{\tilde{w}_{c+1}} = \frac{e^{\lambda_c\left(\tilde{S}_c-\tilde{S}'_c\right)}\tilde{p}\left(\tilde{S}'_c\right)}{\tilde{w}_{c+1}} + \frac{e^{\lambda_c\left(\tilde{S}_c-\tilde{S}'_c\right)}-1}{\lambda_c}\left(\frac{\tilde{w}_c}{\tilde{w}_{c+1}}\right). \tag{36}$$

Let us now show that if $\tilde{p}'\left(\tilde{S}'_c\right)/\tilde{w}'_c \geq \tilde{p}\left(\tilde{S}'_c\right)/\tilde{w}_c$, then

$$\frac{e^{\lambda_{c+1}\left(\tilde{S}_c-\tilde{S}'_c\right)}\tilde{p}'\left(\tilde{S}'_c\right)}{\tilde{w}'_{c+1}} \geq \frac{e^{\lambda_c\left(\tilde{S}_c-\tilde{S}'_c\right)}\tilde{p}\left(\tilde{S}'_c\right)}{\tilde{w}_{c+1}}, \tag{37}$$

$$\frac{e^{\lambda_{c+1}\left(\tilde{S}_c-\tilde{S}'_c\right)}-1}{\lambda_{c+1}} \geq \frac{e^{\lambda_c\left(\tilde{S}_c-\tilde{S}'_c\right)}-1}{\lambda_c}\left(\frac{\tilde{w}_c}{\tilde{w}_{c+1}}\right). \tag{38}$$

We start with inequality (37), which can be rearranged as

$$e^{\lambda_{c+1}\left(\tilde{S}_c-\tilde{S}'_c\right)}\frac{\tilde{w}'_c}{\tilde{w}'_{c+1}}\frac{\tilde{p}'\left(\tilde{S}'_c\right)}{\tilde{w}'_c} \geq e^{\lambda_c\left(\tilde{S}_c-\tilde{S}'_c\right)}\frac{\tilde{w}_c}{\tilde{w}_{c+1}}\frac{\tilde{p}\left(\tilde{S}'_c\right)}{\tilde{w}_c}.$$

By equation (7), we know that

$$e^{\lambda_c\left(\tilde{S}_c-\tilde{S}'_c\right)}\frac{\tilde{w}_c}{\tilde{w}_{c+1}}\frac{\tilde{p}\left(\tilde{S}'_c\right)}{\tilde{w}_c} = \frac{\frac{\tilde{p}\left(\tilde{S}'_c\right)}{\tilde{w}_c}}{1+\left(\lambda_c-\lambda_{c+1}\right)\frac{\tilde{p}\left(\tilde{S}_c\right)}{\tilde{w}_c}}, \tag{39}$$

$$e^{\lambda_{c+1}\left(\tilde{S}_c-\tilde{S}'_c\right)}\frac{\tilde{w}'_c}{\tilde{w}'_{c+1}}\frac{\tilde{p}'\left(\tilde{S}'_c\right)}{\tilde{w}'_c} = \frac{\frac{\tilde{p}'\left(\tilde{S}'_c\right)}{\tilde{w}'_c}}{1+\left(\lambda_c-\lambda_{c+1}\right)\frac{\tilde{p}'\left(\tilde{S}'_c\right)}{\tilde{w}'_c}}. \tag{40}$$

Under the assumption that $\tilde{p}'\left(\tilde{S}'_c\right)/\tilde{w}'_c \geq \tilde{p}\left(\tilde{S}'_c\right)/\tilde{w}_c$, equations (39) and (40) imply

$$e^{\lambda_{c+1}\left(\tilde{S}_c-\tilde{S}'_c\right)}\frac{\tilde{w}'_c}{\tilde{w}'_{c+1}}\frac{\tilde{p}'\left(\tilde{S}'_c\right)}{\tilde{w}'_c} \geq \frac{\frac{\tilde{p}\left(\tilde{S}'_c\right)}{\tilde{w}_c}}{1+\left(\lambda_c-\lambda_{c+1}\right)\frac{\tilde{p}\left(\tilde{S}'_c\right)}{\tilde{w}_c}} > e^{\lambda_c\left(\tilde{S}_c-\tilde{S}'_c\right)}\frac{\tilde{w}_c}{\tilde{w}_{c+1}}\frac{\tilde{p}\left(\tilde{S}'_c\right)}{\tilde{w}_c},$$

where the second inequality also uses the fact that $\tilde{S}'_c < \tilde{S}_c$. Thus inequality (37) holds. Let us now consider inequality (38), which can be rearranged as

$$\frac{\lambda_c\left(e^{\lambda_{c+1}\left(\tilde{S}_c-\tilde{S}_c'\right)}-1\right)}{\lambda_{c+1}\left(e^{\lambda_c\left(\tilde{S}_c-\tilde{S}_c'\right)}-1\right)}\geq\frac{\tilde{w}_c}{\tilde{w}_{c+1}}. \tag{41}$$

Since $\tilde{S}_{c-1}<\tilde{S}_c'$ for any $c<c_0$, condition (2) implies $\tilde{p}_c\left(\tilde{S}_c\right)/\tilde{w}_c\geq\left[e^{\lambda_c\left(\tilde{S}_c-\tilde{S}_{c-1}\right)}-1\right]/\lambda_c>\left[e^{\lambda_c\left(\tilde{S}_c-\tilde{S}_c'\right)}-1\right]/\lambda_c$. Combining the previous inequality with equation (7), we obtain

$$\frac{\tilde{w}_c}{\tilde{w}_{c+1}}=\frac{1}{1+(\lambda_c-\lambda_{c+1})\frac{\tilde{p}\left(\tilde{S}_c\right)}{\tilde{w}_c}}<\frac{\lambda_c}{\lambda_c+(\lambda_c-\lambda_{c+1})\left[e^{\lambda_c\left(\tilde{S}_c-\tilde{S}_c'\right)}-1\right]} \tag{42}$$

By inequalities (41) and (42), a sufficient condition for inequality (38) to hold is

$$\frac{\lambda_c\left[e^{\lambda_{c+1}\left(\tilde{S}_c-\tilde{S}_c'\right)}-1\right]}{\lambda_{c+1}\left[e^{\lambda_c\left(\tilde{S}_c-\tilde{S}_c'\right)}-1\right]}\geq\frac{\lambda_c}{\lambda_c+(\lambda_c-\lambda_{c+1})\left[e^{\lambda_c\left(\tilde{S}_c-\tilde{S}_c'\right)}-1\right]},$$

which can be rearranged as $\lambda_c/\left[1-e^{-\lambda_c\left(\tilde{S}_c-\tilde{S}_c'\right)}\right]\geq\lambda_{c+1}/\left[1-e^{-\lambda_{c+1}\left(\tilde{S}_c-\tilde{S}_c'\right)}\right]$. The previous inequality necessarily holds since $f(x)\equiv\frac{x}{1-e^{-tx}}$ is increasing in x for $t>0$. At this point, we have established that inequalities (37) and (38) hold if $\tilde{p}'\left(\tilde{S}_c'\right)/\tilde{w}_c'\geq\tilde{p}\left(\tilde{S}_c'\right)/\tilde{w}_c$. Combining this observation with equations (35) and (36), we further have that for any $c<c_0$,

$$\tilde{p}'\left(\tilde{S}_c'\right)/\tilde{w}_c'\geq\tilde{p}\left(\tilde{S}_c'\right)/\tilde{w}_c\Rightarrow\tilde{p}'\left(\tilde{S}_c\right)/\tilde{w}_{c+1}'\geq\tilde{p}\left(\tilde{S}_c\right)/\tilde{w}_{c+1}. \tag{43}$$

Since $\tilde{p}'(0)=\tilde{p}(0)=0$, we know that $\tilde{p}'\left(\tilde{S}_0\right)/\tilde{w}_1'\geq\tilde{p}\left(\tilde{S}_0\right)/\tilde{w}_1$. Thus we can use implications (34) and (43) to establish, by iteration, that

$$\frac{\tilde{p}'\left(\tilde{S}_{c_0-1}\right)}{\tilde{w}_{c_0}'}\geq\frac{\tilde{p}\left(\tilde{S}_{c_0-1}\right)}{\tilde{w}_{c_0}}. \tag{44}$$

Since $\widetilde{S}_{c_0}<\widetilde{S}_{c_0}'$, we know from condition (2) that

$$\frac{\tilde{p}'\left(\tilde{S}_{c_0}\right)}{\tilde{w}_{c_0}'}=\frac{e^{\lambda_{c_0}\left(\tilde{S}_{c_0}-\tilde{S}_{c_0-1}\right)}\tilde{p}'\left(\tilde{S}_{c_0-1}\right)}{\tilde{w}_{c_0}'}+\frac{e^{\lambda_{c_0}\left(\tilde{S}_{c_0}-\tilde{S}_{c_0-1}\right)}-1}{\lambda_{c_0}}, \tag{45}$$

$$\frac{\tilde{p}\left(\tilde{S}_{c_0}\right)}{\tilde{w}_{c_0}}=\frac{e^{\lambda_{c_0}\left(\tilde{S}_{c_0}-\tilde{S}_{c_0-1}\right)}\tilde{p}\left(\tilde{S}_{c_0-1}\right)}{\tilde{w}_{c_0}}+\frac{e^{\lambda_{c_0}\left(\tilde{S}_{c_0}-\tilde{S}_{c_0-1}\right)}-1}{\lambda_{c_0}}. \tag{46}$$

Inequality (44) and equations (45) and (46) imply $\tilde{p}'\left(\tilde{S}_{c0}\right)/\tilde{w}'_{c0} \geq \tilde{p}\left(\tilde{S}_{c0}\right)/\tilde{w}_{c0}$. Finally, since $\widetilde{S}_{c0} < \widetilde{S}'_{c0}$, we also know that $\tilde{p}'\left(\tilde{S}'_{c0}\right)/\tilde{w}'_{c0} > \tilde{p}'\left(\tilde{S}_{c0}\right)/\tilde{w}'_{c0}$. Combining these two observations, we get $\tilde{p}'\left(\tilde{S}'_{c0}\right)/\tilde{w}'_{c0} > \tilde{p}\left(\tilde{S}_{c0}\right)/\tilde{w}_{c0}$. Together with equation (7), the previous inequality implies $(\tilde{w}_{c0+1}/\tilde{w}_{c0})' > (\tilde{w}_{c0+1}/\tilde{w}_{c0})$, which contradicts inequality (32). Thus $(w_{c0+1}/w_{c0})' > (w_{c0+1}/w_{c0})$, which implies $c_1 \equiv \inf\left\{c \geq c_0 | (w_{c+1}/w_c)' < (w_{c+1}/w_c)\right\} > c_0$. As mentioned above, Proposition 3 directly follows from Steps 2-4. **QED**. ■

C Proofs (III): Technological Change

Proof of Proposition 4. Like in the proof of Proposition 2, we use equations (23) and (24), which both derive from Lemma 2, and are reported below for expositional purposes:

$$Q_{c-1} = \frac{\lambda_c L_c}{1-e^{-\lambda_c N_c}}, \text{ for all } c \in \mathcal{C} \tag{23}$$

$$Q_c = \frac{\lambda_c L_c e^{-\lambda_c N_c}}{1-e^{-\lambda_c N_c}}, \text{ for all } c \in \mathcal{C}, \tag{24}$$

For future reference, note that $\partial Q_{c-1}/\partial N_c < 0$ and $\partial Q_c/\partial N_c < 0$, whereas $\partial Q_{c-1}/\partial \lambda_c > 0$ and $\partial Q_c/\partial \lambda_c < 0$. The rest of our proof proceeds in six steps.

Step 1: *If* $\lambda'_{c0} < \lambda_{c0}$ *and* $c_0 \neq C$, *then* $N'_C < N_C$.

We first establish that $Q'_C > Q_C$. By the same argument as in Step 1 of Proposition 2, Q'_C must be such that

$$Q'_C = \underset{\widetilde{Q}_1(\cdot),\ldots,\widetilde{Q}_C(\cdot)}{\arg\max} \sum_{c=1}^{C} \widetilde{Q}_c(S),$$

subject to

$$\sum_{c=1}^{C} \widetilde{Q}_c(s_2) - \sum_{c=1}^{C} \widetilde{Q}_c(s_1) \leq -\int_{s_1}^{s_2} \sum_{c=1}^{C} \lambda'_c \widetilde{Q}_c(s)\, ds, \text{ for all } s_1 \leq s_2, \tag{47}$$

$$\int_0^S \widetilde{Q}_c(s)\, ds \leq L_c, \text{ for all } c \in \mathcal{C}, \tag{48}$$

where $\lambda'_{c0} < \lambda_{c0}$ and $\lambda'_c = \lambda_c$ for all $c \neq c_0$. Now consider $\widetilde{Q}_1(\cdot), \ldots, \widetilde{Q}_C(\cdot)$ such that

$$\widetilde{Q}_{c0}(s) \equiv e^{-\left(\lambda_{c0}-\lambda'_{c0}\right)\left(S_{c0}-s\right)} Q_{c0}(s) + \left(\frac{\lambda'_{c0} e^{-\lambda'_{c0} s}}{1-e^{-\lambda'_{c0} S}}\right) \int_{S_{c0-1}}^{S_{c0}} \left[1 - e^{-\left(\lambda_{c0}-\lambda'_{c0}\right)\left(S_{c0}-t\right)}\right] Q_{c0}(t)\, dt, \text{ for all } s \in \mathcal{S},$$

and

$$\widetilde{Q}_c(s) \equiv Q_c(s), \text{ for all } s \in \mathcal{S} \text{ and } c \neq c_0.$$

Since $Q_1(\cdot), \ldots, Q_C(\cdot)$ satisfy the initial resource and technological constraints, as described by equations (3) and (4), $\widetilde{Q}_1(\cdot), ..., \widetilde{Q}_C(\cdot)$ must satisfy, by construction, the new resource and technological constraints, as described by equations (47) and (48). Since $\lambda'_{c_0} < \lambda_{c_0}$, we must also have

$$\widetilde{Q}_{c_0}(1) + \widetilde{Q}_C(1) = \left(\frac{\lambda_{c_0} e^{-\lambda_{c_0} S}}{1 - e^{-\lambda_{c_0} S}}\right) \int_{S_{c_0-1}}^{S_{c_0}} \left[1 - e^{-\left(\lambda_{c_0} - \lambda'_{c_0}\right)\left(S_{c_0} - t\right)}\right] Q_{c_0}(t)\, dt + Q_C > Q_C.$$

Since $Q'_C \geq \widetilde{Q}_{c_0}(1) + \widetilde{Q}_C(1)$, the previous inequality implies $Q'_C > Q_C$. Combining this observation with equation (24) and the fact that $c_0 \neq C$, which implies $\lambda'_C = \lambda_C$, we get $N'_C < N_C$.

Step 2: *If $\lambda'_{c_0} < \lambda_{c_0}$, then $N'_c < N_c$ for all $c > c_0$.*

We proceed by iteration. By Step 1, we know that if $\lambda'_{c_0} < \lambda_{c_0}$ and $c_0 \neq C$, then $N'_C < N_C$. Let us now show that if $N'_c < N_c$ and $c > c_0 + 1$, then $N'_{c-1} < N_{c-1}$. Since $c > c_0 + 1$, we know that $\lambda'_c = \lambda_c$. Thus $N'_c < N_c$ and equation (23) imply $Q'_{c-1} > Q_{c-1}$. Since $c > c_0 + 1$, we also know that $\lambda'_{c-1} = \lambda_{c-1}$. Thus $Q'_{c-1} > Q_{c-1}$ and equation (24) imply $N'_{c-1} < N_{c-1}$. This completes the proof of Step 2.

Step 3. *If $\lambda'_{c_0} < \lambda_{c_0}$ and $c_0 \neq 1$, then $N'_1 > N_1$.*

We proceed by contradiction. Suppose that $N'_1 \leq N_1$. Since $c_0 \neq 1$, we know that $\lambda'_1 = \lambda_1$. Thus $N'_1 \leq N_1$ and equation (24) imply $Q'_1 \geq Q_1$. Let us now show that if $N'_c \leq N_c$ and $Q'_c \geq Q_c$ for $c < c_0 - 1$, then $N'_{c+1} \leq N_{c+1}$ and $Q'_{c+1} \geq Q_{c+1}$. Since $c < c_0 - 1$, we know that $\lambda'_{c+1} = \lambda_{c+1}$. Thus $Q'_c \geq Q_c$ and equation (23) imply $N'_{c+1} \leq N_{c+1}$, whereas $N'_{c+1} \leq N_{c+1}$ and equation (24) imply $Q'_{c+1} \geq Q_{c+1}$. At this point, we have established, by iteration, that if $N'_1 \leq N_1$, then $N'_c \leq N_c$ and $Q'_c \geq Q_c$ for all countries $c < c_0$. Let us now turn to countries $c \geq c_0$. First consider country c_0. By assumption, we know that $\lambda'_{c_0} < \lambda_{c_0}$. Thus $Q'_{c_{0-1}} \geq Q_{c_0-1}$ and equation (23) imply $N'_{c_0} < N_{c_0}$, whereas $N'_{c_0} < N_{c_0}$ and equation (24) imply $Q'_{c_0} > Q_{c_0}$. Using the same logic as before, let us now show that if $N'_c < N_c$ and $Q'_c > Q_c$ for $c \geq c_0$, then $N'_{c+1} < N_{c+1}$ and $Q'_{c+1} > Q_{c+1}$. Since $c \geq c_0$, we know that $\lambda'_{c+1} = \lambda_{c+1}$. Thus $Q'_c > Q_c$ and equation (23) imply $N'_{c+1} < N_{c+1}$, whereas $N'_{c+1} < N_{c+1}$ and equation (24) imply $Q'_{c+1} > Q_{c+1}$. At this point, we have therefore established, by iteration, that $N'_c \leq N_c$ and $Q'_c \geq Q_c$ for all countries $c \in \mathcal{C}$, with strict inequality for all countries $c \geq c_0$. This implies $\sum_{c=1}^{C} N'_c = S'_C - S'_0 < S_C - S_0 = \sum_{c=1}^{C} N_c$, which contradicts $S'_C - S'_0 = S_C - S_0 = 1$ by Lemma 2.

Step 4 *If $\lambda'_{c_0} < \lambda_{c_0}$, then $N'_c > N_c$ for all $c < c_0$.*

We proceed by iteration. By Step 3, we know that if $\lambda'_{c_0} < \lambda_{c_0}$ and $c_0 \neq 1$, then $N'_1 > N_1$. Since $c_0 \neq 1$, we know that $\lambda'_1 = \lambda_1$. Thus $N'_1 > N_1$ and equation (24) imply $Q'_1 < Q_1$. Let us now show that if $N'_c > N_c$ and $Q'_c < Q_c$ for $c < c_0 - 1$, then $N'_{c+1} > N_{c+1}$ and $Q'_{c+1} < Q_{c+1}$. Since $c < c_0 - 1$, we know that $\lambda'_{c+1} = \lambda_{c+1}$. Thus $Q'_c < Q_c$ and equation (23) imply $N'_{c+1} > N_{c+1}$, whereas $N'_{c+1} > N_{c+1}$ and equation (24) imply $Q'_{c+1} < Q_{c+1}$. This completes the proof of Step 4.

Step 5: *If $\lambda'_{c_0} < \lambda_{c_0}$, then $S'_c \geq S_c$ for all $c \geq c_0$.*

The proof is identical to the proof of Step 5 in Proposition 2 and omitted.

Step 6: *If $\lambda'_{c_0} < \lambda_{c_0}$, then $S'_{c-1} \geq S_{c-1}$ for all $c \leq c_0$.*

We proceed by iteration. By Lemma 2, we know that $S'_0 = S_0 = 0$. Thus $S'_{c-1} \geq S_{c-1}$ is satisfied for $c = 1$. Let us now show that if $S'_{c-1} \geq S_{c-1}$ and $c < c_0$, then $S'_c \geq S_c$. Since $c < c_0$, $L'_{c_0} > L_{c_0}$ and Step 4 imply $N'_c > N_c$. Combining this observation with $S'_{c-1} \geq S_{c-1}$ and the definition of $N_c \equiv S_c - S_{c-1}$, we obtain $S'_c \geq S_c$. This completes the proof of Step 6.

To conclude, note that Steps 2 and 4 imply that $N'_c > N_c$ for all $c < c_0$ and $N'_c < N_c$ for all $c > c_0$. Finally, Definition 2, Step 5, and Step 6 imply that all countries $c \neq c_0$ move up the supply chain. **QED.** ■

Proof of Proposition 5. Following the same logic as in the proof Proposition 3, one can show the following lemma.

Lemma 5 *For any country* $1 < c < C$, *if* $N'_c > N_c$ *and* $(w_c/w_{c-1})' > (w_c/w_{c-1})$, *then* $(w_{c+1}/w_c)' > (w_{c+1}/w_c)$.

Proof of Lemma 5. The proof is similar to the proof of Lemma 4 and omitted. ■

Proof of Proposition 5 (Continued). The rest of the proof of Proposition 5 proceeds in three steps.

Step 1: *If* $\lambda'_{c_0} < \lambda_{c_0}$ *and* $c_0 \neq 1$, *then* $(w_2/w_1)' > (w_2/w_1)$.

Like in the proof of Proposition 3, Equation (7), equation (8), and $p_0 = 0$ imply

$$\frac{w_2}{w_1} = 1 + \frac{1}{\lambda_1} (\lambda_1 - \lambda_2) \left(e^{\lambda_1 N_1} - 1 \right). \tag{49}$$

By Proposition 4, we know that $\lambda'_{c_0} < \lambda_{c_0}$ implies $N'_1 > N_1$. Since $c_0 \neq 1$, we also know that $\lambda'_1 = \lambda_1$. Combining the two previous observations with equation (31), we obtain $(w_2/w_1)' > (w_2/w_1)$.

Step 2: $(w_{c+1}/w_c)' > (w_{c+1}/w_c)$ *for all* $c < c_0$.

By Proposition 4, we know that $\lambda'_{c_0} < \lambda_{c_0}$ implies $N'_c > N_c$ for all $c < c_0$. Therefore, we can invoke Step 1 and Lemma 5 to establish, by induction, that $(w_{c+1}/w_c)' > (w_{c+1}/w_c)$ for all $c < c_0$.

Step 3: *If there exists* $\widetilde{c} > c_0$ *such that* $(w_{\widetilde{c}+1}/w_{\widetilde{c}})' < (w_{\widetilde{c}+1}/w_{\widetilde{c}})$, *then* $(w_{c+1}/w_c)' < (w_{c+1}/w_c)$ *for all* $\widetilde{c} \leq c < C$.

Suppose that there exists $\widetilde{c} > c_0$ such that $(w_{\widetilde{c}+1}/w_{\widetilde{c}})' < (w_{\widetilde{c}+1}/w_{\widetilde{c}})$. By Proposition 4, we know that $\lambda'_{c_0} < \lambda_{c_0}$ implies $N'_c < N_c$ for all $c > \widetilde{c} > c_0$. Therefore, we can use the same strategy as in Lemma 4 in the proof of Proposition 3 to establish that $(w_{c+1}/w_c)' < (w_{c+1}/w_c)$ for all $\widetilde{c} \leq c < C$. This concludes the proof of Step 3.

To conclude, let us define $c_1 \equiv \inf \left\{ c \geq c_0 | \, (w_{c+1}/w_c)' > (w_{c+1}/w_c) \right\}$, and similarly, let us define $c_2 \equiv \inf \left\{ c > c_1 | \, (w_{c+1}/w_c)' < (w_{c+1}/w_c) \right\}$. Step 2 implies that if $\lambda'_{c_0} < \lambda_{c_0}$, then w_{c+1}/w_c rises for all $c < c_0$. The definitions of c_1 and c_2 imply that w_{c+1}/w_c falls for all $c_0 \leq c < c_1$ and rises for all $c_1 \leq c < c_2$. Step 3 and the definition of c_2 imply that w_{c+1}/w_c falls for all $c_2 \leq c < C$. Proposition 5 directly derives from the previous observations and Definition 3. **QED.** ■